

# A Poisson-Jacobi-type transformation for the sum $\sum_{n=1}^{\infty} n^{-2m} \exp(-an^2)$ for positive integer $m$

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## Abstract

We obtain an asymptotic expansion for the sum

$$S(a; w) = \sum_{n=1}^{\infty} \frac{e^{-an^2}}{n^w}$$

as  $a \rightarrow 0$  in  $|\arg a| < \frac{1}{2}\pi$  for arbitrary finite  $w > 0$ . The result when  $w = 2m$ , where  $m$  is a positive integer, is the analogue of the well-known Poisson-Jacobi transformation for the sum with  $m = 0$ . Numerical results are given to illustrate the accuracy of the expansion.

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## 1. Introduction

The classical Poisson-Jacobi transformation is given by

$$\sum_{n=1}^{\infty} e^{-an^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} - \frac{1}{2} + \sqrt{\frac{\pi}{a}} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 / a}, \quad (1.1)$$

where the parameter  $a$  satisfies  $\Re(a) > 0$ . This transformation relates a sum of Gaussian exponentials involving the parameter  $a$  to a similar sum with parameter  $\pi^2/a$ . In the case  $a \rightarrow 0$  in  $\Re(a) > 0$ , the convergence of the sum on the left-hand side becomes slow, whereas the sum on the right-hand side converges rapidly in this limit. Various proofs of the well-known result (1.1) exist in the literature; see, for example, [3, p. 120], [4, p. 60] and [5, p. 124].

In this note we consider the sum

$$S(a; w) = \sum_{n=1}^{\infty} \frac{e^{-an^2}}{n^w} \quad (\Re(a) > 0). \quad (1.2)$$

This sum converges for any finite value of the parameter  $w$  provided  $\Re(a) > 0$ ; when  $a = 0$  then  $S(0; w)$  reduces to the Riemann zeta function  $\zeta(w)$  when  $\Re(w) > 1$ . Consequently, the series in (1.2) can be viewed as a smoothed Dirichlet series for  $\zeta(w)$ . The asymptotic expansion

of  $S(a; w)$  as  $a \rightarrow 0$  in  $\Re(a) > 0$  is straightforward. The most interesting case arises when  $w = 2m$ , where  $m$  is a positive integer, for which we establish a transformation for  $S(a; 2m)$  analogous to that in (1.1) valid as  $a \rightarrow 0$  in  $\Re(a) > 0$ . This similarly involves the series in (1.2) with  $a$  replaced by  $\pi^2/a$ , but with each term decorated by an asymptotic series in  $a$ . A recent application of the series with  $w = 2$  and  $w = 4$  has arisen in the geological problem of thermochronometry in spherical geometry [6].

## 2. An expansion for $S(a; w)$ as $a \rightarrow 0$ when $w \neq 2, 4, \dots$

Our starting point is the well-known Cahen-Mellin integral (see, for example, [3, §3.3.1])

$$z^{-\alpha} e^{-z} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Gamma(s-\alpha) z^{-s} ds \quad (z \neq 0, |\arg z| < \tfrac{1}{2}\pi), \quad (2.1)$$

where  $c > \Re(\alpha)$  so that the integration path passes to the right of all the poles of  $\Gamma(s-\alpha)$  situated at  $s+\alpha-k$  ( $k = 0, 1, 2, \dots$ ). For simplicity in presentation we shall assume throughout real values of  $w > 0$ . Then, it follows that

$$\begin{aligned} S(a; w) &= \sum_{n=1}^{\infty} \frac{e^{-an^2}}{n^w} = \sum_{n=1}^{\infty} \frac{n^{-w}}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Gamma(s) (an^2)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Gamma(s) \zeta(2s+w) a^{-s} ds, \end{aligned}$$

upon reversal of the order of summation and integration, which is justified when  $c > \max\{0, \frac{1}{2} - \frac{1}{2}w\}$ , and evaluation of the inner sum in terms of the Riemann zeta function. The integrand possesses simple poles at  $s = \frac{1}{2} - \frac{1}{2}w$  and  $s = -k$  ( $k = 0, 1, 2, \dots$ ), except if  $w = 2m+1$  is an odd positive integer when the pole at  $s = \frac{1}{2} - \frac{1}{2}w$  is double. The case when  $w = 2m$  is an even positive integer requires a separate investigation which is discussed in Section 3.

Consider the integral taken round the rectangular contour with vertices at  $c \pm iT$ ,  $-c' \pm iT$ , where  $c' > 0$ . The contribution from the upper and lower sides  $s = \sigma \pm iT$ ,  $-c' \leq \sigma \leq c$ , vanishes as  $T \rightarrow \infty$  provided  $|\arg a| < \frac{1}{2}\pi$ , since from the behaviour

$$\Gamma(\sigma \pm it) = O(t^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\pi t}), \quad \zeta(\sigma \pm it) = O(t^{\mu(\sigma)} \log^A t), \quad (t \rightarrow \infty),$$

where for  $\sigma$  and  $t$  real

$$\mu(\sigma) = 0 \quad (\sigma > 1), \quad \frac{1}{2} - \frac{1}{2}\sigma \quad (0 \leq \sigma \leq 1), \quad \frac{1}{2} - \sigma \quad (\sigma < 0),$$

$$A = 1 \quad (0 \leq \sigma \leq 1), \quad A = 0 \text{ otherwise},$$

the modulus of the integrand is controlled by  $O(T^{\sigma+\mu(\sigma)-\frac{1}{2}} \log T e^{-\Delta T})$ , with  $\Delta = \frac{1}{2}\pi - |\arg a|$ . The residue at the double pole  $s = -m$  when  $w = 2m+1$  ( $m = 0, 1, 2, \dots$ ) is given by

$$\frac{(-a)^m}{m!} \left\{ \gamma - \frac{1}{2} \log a + \frac{1}{2} \psi(m+1) \right\},$$

where  $\gamma$  is Euler's constant and  $\psi(x)$  is the logarithmic derivative of the gamma function. Displacement of the integration path to the left over the poles then yields (provided  $w \neq 2m$ )

$$S(a; w) = J(a; w) + \sum_{k=0}^{N-1} \frac{(-)^k}{k!} \zeta(w-2k) a^k + R_N, \quad (2.2)$$

where

$$J(a; w) = \begin{cases} \frac{1}{2}\Gamma(\frac{1}{2} - \frac{1}{2}w)a^{(w-1)/2} & (w \neq 2m+1) \\ \frac{(-a)^m}{m!} \left\{ \gamma - \frac{1}{2} \log a + \frac{1}{2} \psi(m+1) \right\} & (w = 2m+1), \end{cases}$$

$N$  is a positive integer such that  $N > \frac{1}{2}w + \frac{1}{2}$  and the prime on the sum over  $k$  denotes the omission of the term corresponding to  $k = m$  when  $w = 2m+1$ .

The remainder  $R_N$  is

$$R_N = \frac{1}{2\pi i} \int_{-c-\infty i}^{-c+\infty i} \Gamma(s) \zeta(w+2s) a^{-s} ds, \quad c = N - \frac{1}{2}. \quad (2.3)$$

It is shown in the appendix, when  $w \neq 2, 4, \dots$ , that  $R_N = O(a^{N-\frac{1}{2}})$  as  $a \rightarrow 0$  in  $|\arg a| < \frac{1}{2}\pi$ , with the constant implied in the  $O$ -symbol growing at least like  $\Gamma(N+1-\frac{1}{2}w)$ . This establishes that the above series over  $k$  diverges as  $N \rightarrow \infty$  and that (2.2) is therefore an asymptotic expansion.

We remark that the algebraic expansion (2.2) also contains a subdominant exponentially small component as  $a \rightarrow 0$ ; compare [3, §8.1.5] for the particular case  $w = 0$ . We do not consider this further in the present paper.

### 3. An expansion for $S(a; 2m)$ when $m = 1, 2, \dots$

The case  $w = 2m$ , where  $m$  is a positive integer, is more interesting as this leads to the analogue of the Poisson-Jacobi transformation (1.1). There is now only a finite set of poles of the integrand in (2.1) at  $s = \frac{1}{2} - \frac{1}{2}w$  and  $s = 0, -1, -2, \dots, -m$ , since the poles of  $\Gamma(s)$  at  $s = -m - k$  ( $k = 1, 2, \dots$ ) are cancelled by the trivial zeros of the zeta function  $\zeta(2m+2s)$  at  $s = -m - 1, -m - 2, \dots$ . This has the consequence that the integrand is holomorphic in  $\Re(s) < -m$ , so that further displacement of the contour can produce no additional algebraic terms in the expansion of  $S(a; 2m)$ . Thus, we find when  $w = 2m$

$$S(a; 2m) = \frac{1}{2}\Gamma(\frac{1}{2} - m)a^{m-\frac{1}{2}} + \sum_{k=0}^m \frac{(-)^k}{k!} \zeta(2m-2k) a^k + I_L, \quad (3.1)$$

where, upon making the change of variable  $s \rightarrow -s$ ,

$$I_L = \frac{1}{2\pi i} \int_L \Gamma(-s) \zeta(2m-2s) a^s ds \quad (3.2)$$

and  $L$  denotes a path parallel to the imaginary axis with  $\Re(s) > m$ .

We now employ the functional relation for  $\zeta(s)$  given by [5, p. 269]

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin \frac{1}{2}\pi s \quad (3.3)$$

to convert the argument of the zeta function in (3.2) into one with real part greater than unity. The integral in (3.2) can then be written in the form

$$\frac{(-)^m (2\pi)^{2m}}{2\pi i} \int_L \zeta(2s-2m+1) \frac{\Gamma(2s-2m+1)}{\Gamma(s+1)} \left(\frac{a}{4\pi^2}\right)^s ds.$$

Since on the integration path  $\Re(2s-2m+1) > 1$ , we can expand the zeta function and reverse the order of summation and integration to obtain

$$I_L = (-)^m \pi^{2m-\frac{1}{2}} \sum_{n=1}^{\infty} n^{2m-1} K_n(a; m), \quad (3.4)$$

where

$$K_n(a; m) := \frac{1}{2\pi i} \int_L \frac{\Gamma(s - m + \frac{1}{2})\Gamma(s - m + 1)}{\Gamma(s + 1)} \left(\frac{a}{\pi^2 n^2}\right)^s ds,$$

and we have employed the duplication formula for the gamma function

$$\Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2}).$$

The integrals  $K_n(a; m)$  have no poles in the half-plane  $\Re(s) > m$ , so that we can displace the path  $L$  as far to the right as we please. On such a displaced path  $|s|$  is everywhere large. The quotient of gamma functions may then be expanded by making use of the result given in [3, p. 53]

$$\frac{\Gamma(s - m + \frac{1}{2})\Gamma(s - m + 1)}{\Gamma(s + 1)} = \sum_{j=0}^{M-1} (-)^j c_j \Gamma(s + \vartheta - j) + \rho_M(s) \Gamma(s + \vartheta - M) \quad (3.5)$$

for positive integer  $M$ , where  $\vartheta = \frac{1}{2} - 2m$ ,

$$c_j = \frac{(m)_j (m + \frac{1}{2})_j}{j!} = \frac{2^{-2j} (2m)_{2j}}{j!}$$

and  $\rho_M(s) = O(1)$  as  $|s| \rightarrow \infty$  in  $|\arg s| < \pi$ . Substitution of this expansion into the integrals  $K_n(a; m)$  then produces

$$\begin{aligned} K_n(a; m) &= \sum_{j=0}^{M-1} (-)^j c_j \frac{1}{2\pi i} \int_L \Gamma(s + \vartheta - j) \left(\frac{a}{\pi^2 n^2}\right)^s ds + \mathcal{R}_M \\ &= \sum_{j=0}^{M-1} (-)^j c_j \left(\frac{a}{\pi^2 n^2}\right)^{2m+j-\frac{1}{2}} e^{-\pi^2 n^2/a} + \mathcal{R}_M \end{aligned} \quad (3.6)$$

by (2.1), where

$$\mathcal{R}_M = \frac{1}{2\pi i} \int_L \rho_M(s) \Gamma(s + \vartheta - M) \left(\frac{a}{\pi^2 n^2}\right)^s ds.$$

Bounds for the remainder  $\mathcal{R}_M$  have been considered in [3, p. 71, Lemma 2.7], where it is shown that

$$\mathcal{R}_M = O\left(\left(\frac{a}{\pi^2 n^2}\right)^{M-\vartheta} e^{-\pi^2 n^2/a}\right) \quad (3.7)$$

as  $a \rightarrow 0$  in the sector  $|\arg a| < \frac{1}{2}\pi$ .

Collecting together the results in (3.2), (3.4), (3.6) and (3.7), we obtain

$$I_L = (-)^m \left(\frac{a}{\pi}\right)^{2m-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{e^{-\pi^2 n^2/a}}{n^{2m}} \left\{ \sum_{j=0}^{M-1} c_j \left(\frac{-a}{\pi^2 n^2}\right)^j + O\left(\left(\frac{a}{\pi^2 n^2}\right)^M\right) \right\}.$$

From (3.1) we now have the following theorem:

**Theorem 1.** *Let  $m$  and  $M$  be positive integers. Then, when  $w = 2m$ , we have the expansion valid as  $a \rightarrow 0$  in  $|\arg a| < \frac{1}{2}\pi$*

$$S(a; 2m) = \frac{1}{2} \Gamma\left(\frac{1}{2} - m\right) a^{m-\frac{1}{2}} + \sum_{k=0}^m \frac{(-)^k}{k!} \zeta(2m - 2k) a^k$$

$$+(-)^m \left(\frac{a}{\pi}\right)^{2m-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\Upsilon_n(a; m)}{n^{2m}} e^{-\pi^2 n^2/a}, \quad (3.8)$$

where  $\Upsilon_n(a; m)$  has the asymptotic expansion

$$\Upsilon_n(a; m) = \sum_{j=0}^{M-1} \frac{(m)_j (m + \frac{1}{2})_j}{j!} \left(\frac{-a}{\pi^2 n^2}\right)^j + O\left(\left(\frac{a}{\pi^2 n^2}\right)^M\right).$$

This is the analogue of the Poisson-Jacobi transformation in (1.1). In the case  $m = 0$ , the quotient of gamma functions in (3.5) is replaced by the single gamma function  $\Gamma(s + \frac{1}{2})$ , with the result that  $c_0 = 1$ ,  $c_j = 0$  ( $j \geq 1$ ) and  $\Upsilon_n(a; m) = 1$  for all  $n \geq 1$ . Then (3.8) reduces to (1.1) and is valid for all values of the parameter  $a$  (not just  $a \rightarrow 0$ ) satisfying  $|\arg a| < \frac{1}{2}\pi$ .

**Remark 1.** We note that the values of the zeta function appearing in (3.8) can be expressed alternatively in terms of Bernoulli numbers by the result [2, p. 605]

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|.$$

#### 4. Numerical results and concluding remarks

From the well-known values [2, p. 605]

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90},$$

we obtain from Theorem 1 the expansions in the cases  $m = 1$  and  $m = 2$  given by

$$S(a; 2) = \frac{\pi^2}{6} + \frac{a}{2} - (\pi a)^{\frac{1}{2}} - \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{e^{-\pi^2 n^2/a}}{n^2} \left\{ \sum_{j=0}^{M-1} \left(\frac{3}{2}\right)_j \left(\frac{-a}{\pi^2 n^2}\right)^j + O(a^M) \right\} \quad (4.1)$$

and

$$S(a; 4) = \frac{\pi^4}{90} - \frac{\pi^2 a}{6} - \frac{a^2}{4} + \frac{2}{3} \pi^{\frac{1}{2}} a^{\frac{3}{2}} + \left(\frac{a}{\pi}\right)^{\frac{7}{2}} \sum_{n=1}^{\infty} \frac{e^{-\pi^2 n^2/a}}{n^4} \left\{ \sum_{j=0}^{M-1} \frac{(\frac{5}{2})_j (2)_j}{j!} \left(\frac{-a}{\pi^2 n^2}\right)^j + O(a^M) \right\} \quad (4.2)$$

valid as  $a \rightarrow 0$  in  $|\arg a| < \frac{1}{2}\pi$ .

In Table 1 we show the results of numerical calculations for the case  $m = 2$ . For different values of the parameter  $a$  we present the value of the absolute error in the computation of  $S(a; 4)$  from (4.2). In the computations, we have used only the  $n = 1$  term (since the order of  $\frac{1}{4} \exp(-4\pi^2/a)$  was found to be less than the error), with the expansion for  $\Upsilon_1(a; 2)$  optimally truncated (corresponding to truncation at, or near, the least term in modulus) at index  $j_0 \simeq (\pi^2/a) - \frac{5}{2}$ . It is seen that the error when  $a = 0.1$  is extremely small and that, only when  $a \simeq 2$  does the relative error start to become significant.

To conclude, we mention that a similar treatment can be carried out for the sum

$$S_p(a; w) \equiv \sum_{n=1}^{\infty} \frac{e^{-an^p}}{n^w} \quad (a \rightarrow 0, \Re(a) > 0)$$

Table 1: Values of the absolute error in the computation of  $S(a; 4)$  from (4.2). The value of the index  $j_0$  corresponds to optimal truncation of the expansion  $\Upsilon_1(a; 2)$ .

$a$	$S(a; 4)$	Error	$j_0$
0.10	0.952696	$9.662 \times 10^{-86}$	96
0.20	0.849025	$9.768 \times 10^{-43}$	46
0.25	0.803169	$4.045 \times 10^{-34}$	36
0.50	0.615128	$7.769 \times 10^{-17}$	17
0.75	0.475493	$4.656 \times 10^{-11}$	10
1.00	0.369026	$3.642 \times 10^{-8}$	6
1.50	0.223285	$2.856 \times 10^{-5}$	3
2.00	0.135356	$7.500 \times 10^{-4}$	1

for positive even integer  $w$  and  $p$ . The case  $w = 0$  and  $p > 0$ , corresponding to the Euler-Jacobi series, has been considered in [3, §8.1]; see also [1] for a hypergeometric approach when  $p$  is a rational fraction. The details of the small- $a$  expansion of  $S_p(a; w)$  will be presented elsewhere.

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#### Appendix: A bound for the remainder $R_N$

Let  $\psi = \arg a$  and integer  $N > \frac{1}{2}w + \frac{1}{2}$ . Upon replacement of  $s$  by  $-s$  followed by use of (3.3), the remainder  $R_N$  in (2.3) becomes

$$R_N = \frac{(2\pi)^w}{2\pi i} \int_{N-\frac{1}{2}-\infty i}^{N-\frac{1}{2}+\infty i} \zeta(1-w+2s) \frac{\Gamma(1-w+2s)}{\Gamma(1+s)} \frac{\sin \pi(s-\frac{1}{2}w)}{\sin \pi s} \left(\frac{a}{4\pi^2}\right)^s ds.$$

With  $s = N - \frac{1}{2} + it$ ,  $t \in (-\infty, \infty)$  we have

$$|R_N| \leq (2\pi)^{w-1} \left(\frac{a}{4\pi^2}\right)^{N-\frac{1}{2}} \zeta(2N-w) \int_{-\infty}^{\infty} e^{-\psi t} \left| \frac{\Gamma(2N-w+2it)}{\Gamma(N+\frac{1}{2}+it)} \right| dt,$$

since  $|\zeta(x+it)| \leq \zeta(x)$  ( $x > 1$ ) and

$$\left| \frac{\sin \pi(N - \frac{1}{2} - \frac{1}{2}w + it)}{\sin \pi(N + \frac{1}{2} + it)} \right| = \frac{|\cos \pi(\frac{1}{2}w - it)|}{\cosh \pi t} = \frac{(\cos^2 \frac{1}{2}\pi w + \sinh^2 \pi t)^{\frac{1}{2}}}{\cosh \pi t} \leq 1.$$

It then follows that

$$|R_N| = O\left(\left(\frac{a}{\pi^2}\right)^{N-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\psi t} |\Gamma(N - \frac{1}{2}w + it)| \left| \frac{\Gamma(N - \frac{1}{2}w + \frac{1}{2} + it)}{\Gamma(N + \frac{1}{2} + it)} \right| dt\right). \quad (\text{A.1})$$

Using the argument presented in [3, p. 126], we set  $N - \frac{1}{2}w - \frac{1}{2} = M + \delta$ , with  $-\frac{1}{2} < \delta \leq \frac{1}{2}$  so that  $M \leq N - 1$ , to find

$$\left| \frac{\Gamma(N - \frac{1}{2}w + \frac{1}{2} + it)}{\Gamma(N + \frac{1}{2} + it)} \right| = P(t)g(t), \quad g(t) := \left| \frac{\Gamma(1 + \delta + it)}{\Gamma(\frac{1}{2} + it)} \right|,$$

where

$$P(t) = \left(\frac{1}{4} + t^2\right)^{-\frac{1}{2}} \frac{\prod_{r=1}^M \{(r + \delta)^2 + t^2\}^{\frac{1}{2}}}{\prod_{r=1}^{N-1} \{(r + \frac{1}{2})^2 + t^2\}^{\frac{1}{2}}} \leq \left(\frac{1}{4} + t^2\right)^{-\frac{1}{2}} \leq 2.$$

From the upper bound for the gamma function  $\Gamma(z)$  with  $z = x + it$ ,  $x > 0$  [3, p. 35]

$$\begin{aligned} |\Gamma(z)| &< \Gamma(x)(1 + t^2/x^2)^{\frac{1}{2}x - \frac{1}{4}} e^{-|t|\phi(t)} e^{1/(6|z|)}, \quad \phi(t) = \arctan(|t|/x) \\ &< \Gamma(x)(1 + \tau^2)^{\frac{1}{2}x - \frac{1}{4}} \exp[x\{\omega(\tau) - \frac{1}{2}\pi|\tau|\}] e^{1/(6x)} \\ &< e^x \Gamma(x)(1 + \tau^2)^{\frac{1}{2}x - \frac{1}{4}} e^{-\frac{1}{2}\pi x|\tau|} e^{1/(6x)}, \end{aligned}$$

where we have put  $\tau = t/x$ , defined  $\omega(\tau) = |\tau| \arctan(1/|\tau|)$  and used the fact that  $0 \leq \omega(\tau) < 1$  for  $\tau \in [0, \infty)$ , with the limit 1 being approached as  $\tau \rightarrow \infty$ . Substituting the above bounds into (A.1), we see on setting  $x = N - \frac{1}{2}w$  that

$$|R_N| = e^N \Gamma(N - \frac{1}{2}w + 1) O\left(\left(\frac{a}{\pi^2}\right)^{N - \frac{1}{2}} \int_0^\infty (1 + \tau^2)^{N/2} g(\tau) \{e^{-\Delta_+ \tau} + e^{-\Delta_- \tau}\} d\tau\right),$$

where  $\Delta_\pm = (N - \frac{1}{2}w)(\frac{1}{2}\pi \pm \psi)$ . Since  $g(\tau) = O(\tau^{\delta + \frac{1}{2}})$  as  $\tau \rightarrow \infty$ , the integral is convergent provided  $|\psi| < \frac{1}{2}\pi$  and is manifestly an increasing function of  $N$ .

Hence

$$R_N = O(a^{N - \frac{1}{2}}) \quad (a \rightarrow 0, |\arg a| < \frac{1}{2}\pi), \quad (\text{A.2})$$

with the constant implied in the  $O$ -symbol growing at least like  $\Gamma(N + 1 - \frac{1}{2}w)$  as  $N$  increases.

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